

# FAITHFUL COMPLETELY REDUCIBLE REPRESENTATIONS OF MODULAR LIE ALGEBRAS

DONALD W. BARNES

**ABSTRACT.** Let  $L$  be a Lie algebra of dimension  $n$  over a field  $F$  of characteristic  $p > 0$ . I prove the existence of a faithful completely reducible  $L$ -module of dimension less than or equal to  $p^{n^2-1}$ .

## 1. INTRODUCTION

Let  $L$  be a Lie algebra of dimension  $n$  over the field  $F$ . The Ado-Iwasawa Theorem asserts that there exists a faithful finite-dimensional  $L$ -module  $V$ . There are several extensions of this result which assert the existence of such a module  $V$  with various additional properties. See, for example, Hochschild [3], Barnes [1]. Of importance for this paper is Jacobson's Theorem, [4, Theorem 5.5.2] that every finite-dimensional Lie algebra  $L$  over a field  $F$  of characteristic  $p > 0$  has a finite-dimensional faithful completely reducible module  $V$ . None of these results sets a bound to the dimension of  $V$ , unlike the Leibniz algebra analogue [2] which asserts for a Leibniz algebra of dimension  $n$ , the existence of a faithful Leibniz module of dimension less than or equal to  $n+1$ . This raises the question "Is there an analogous strengthening of the Ado-Iwasawa Theorem?" that is, "For a field  $F$ , does there exist a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that every Lie algebra of dimension  $n$  over  $F$  has a faithful module of dimension less than or equal to  $f(n)$ ?" The main purpose of this paper is to prove the following strengthening of Jacobson's Theorem, thereby answering this question in the affirmative for fields  $F$  of non-zero characteristic.

**Theorem 1.1.** *Let  $F$  be a field of characteristic  $p > 0$  and let  $L$  be a Lie algebra of dimension  $n$  over  $F$ . Then  $L$  has a faithful completely reducible module  $V$  with  $\dim(V) \leq p^{n^2-1}$ .*

In all that follows,  $F$  is a field of characteristic  $p > 0$  and  $L$  is a Lie algebra of dimension  $n$  over  $F$ .

## 2. RESTRICTED LIE ALGEBRAS.

A restricted Lie algebra (see [4, Chapter 2]) is a Lie algebra  $L$  together with a  $p$ -operation, that is, a map  $[p] : L \rightarrow L$  such that for  $a, b \in L$  and  $\lambda \in F$ , we have  $\text{ad}(a^{[p]}) = \text{ad}(a)^p$ ,  $(\lambda a)^{[p]} = \lambda^p a^{[p]}$  and

$$(a + b)^{[p]} = a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a, b),$$

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where the  $s_i(a, b)$  are defined by

$$(\text{ad}(a \otimes X + b \otimes 1))^{p-1}(a \otimes 1) = \sum_{i=1}^{p-1} i s_i(a, b) \otimes X^{i-1}$$

in  $L \otimes_F F[X]$ .

For convenience of reference, we list here some properties of  $p$ -operations.

**Lemma 2.1.** *Let  $(L, [p])$  be restricted Lie algebra. Then*

- (1) *If  $[a, b] = 0$ , then  $[a^{[p]^r}, b^{[p]^s}] = 0$ . In particular,  $[a^{[p]^r}, a^{[p]^s}] = 0$ .*
- (2) *If  $[a, b] = 0$ , then  $(a + b)^{[p]} = a^{[p]} + b^{[p]}$ .*
- (3) *For all  $a, b \in L$ , we have  $s_i(a, b) \in L'$ .*

*Proof.* (1) Since  $\text{ad}(a)b = 0$ ,  $\text{ad}(a)^{p^r}b = 0$ , that is,  $[a^{[p]^r}, b] = 0$ . But  $[b, a^{[p]^r}] = 0$  implies that  $[b^{[p]^s}, a^{[p]^r}] = 0$ .

(2) Since  $\text{ad}(a \otimes X + b \otimes 1)(a \otimes 1) = 0$ , we have  $s_i(a, b) = 0$  for all  $i$ .

(3) Follows immediately from the definition. (In fact, by [4, Lemma 2.1.2],  $s_i(a, b) \in L^p$ .)  $\square$

**Lemma 2.2.** *Let  $(L, [p])$  be a restricted Lie algebra and let  $A$  be an abelian ideal of  $L$ . Then there exists a  $p$ -operation  $[p]'$  on  $L$  such that  $a^{[p]'} = 0$  for all  $a \in A$ .*

*Proof.* Take a basis  $\{a_1, \dots, a_r\}$  of  $A$  and extend this to a basis  $\{a_1, \dots, a_n\}$  of  $L$ . Put  $b_i = a_i^{[p]}$ . For  $i = 1, \dots, r$ , we have  $\text{ad}(a_i)^2 = 0$ . We replace these  $b_i$  with 0. By Jacobson's Theorem [4, Theorem 2.2.3], there exists a  $p$ -operation  $[p]'$  on  $L$  such that  $a_i^{[p]'} = 0$  for  $i = 1, \dots, r$  (and  $a_j^{[p]'} = b_j$  for  $j > r$ ). From Lemma 2.1(2), it then follows that  $a^{[p]'} = 0$  for all  $a \in A$ .  $\square$

**Corollary 2.3.** *Let  $(L, [p])$  be a restricted Lie algebra. Then there exists a  $p$ -operation  $[p]'$  on  $L$  such that every abelian minimal ideal of  $L$  is a  $[p]'$ -ideal.*

*Proof.* The abelian socle  $\text{ASoc}(L)$  is the sum of all the abelian minimal ideals of  $L$ . It is an abelian ideal. By Lemma 2.2, there exists a  $p$ -operation  $[p]'$  which is zero on  $\text{ASoc}(L)$ , and so, on every abelian minimal ideal. Thus every abelian minimal ideal is a  $[p]'$ -ideal.  $\square$

**Theorem 2.4.** *Let  $(L, [p])$  be a restricted Lie algebra of dimension  $n$  over the field  $F$  of characteristic  $p$ . Then  $L$  has a faithful completely reducible module of dimension less than or equal to  $p^{n-1}$ .*

*Proof.* By Corollary 2.3, we may suppose that every abelian minimal ideal is a  $[p]$ -ideal. The result holds for  $n = 1$ . We use induction over  $n$ .

Suppose that  $A_1, A_2$  are distinct minimal  $[p]$ -ideals of  $L$ . Then  $L/A_i$  is a restricted Lie algebra. Since  $\dim(L/A_i) \leq n - 1$ ,  $L/A_i$  has a faithful completely reducible module  $V_i$  with  $\dim(V_i) \leq p^{n-2}$ . But  $V_1 \oplus V_2$  is a faithful completely reducible  $L$ -module and  $\dim(V_1 \oplus V_2) \leq 2p^{n-2} \leq p^{n-1}$ .

Suppose that  $A$  is the only minimal  $[p]$ -ideal of  $L$ . Let  $B \subseteq A$  be a minimal ideal of  $L$ . The representation of  $L$  on  $B$  is a  $[p]$ -representation and its kernel  $K$  is a  $[p]$ -ideal. Either  $K = 0$  or  $K \supseteq A$ . If  $K = 0$ , then  $B$  is a faithful completely reducible  $L$ -module and the result holds. So we may suppose that  $K \supseteq A$ . But this implies that  $B$  is abelian. By our choice of  $[p]$ , this implies that  $B$  is a  $[p]$ -ideal and so, that  $B = A$ . Hence we may assume that our only minimal  $[p]$ -ideal  $A$  is also a minimal ideal and is abelian.

We can take a linear map  $c : L \rightarrow F$  such that  $c(A) \neq 0$ . Let  $W = \langle w \rangle$  be the 1-dimensional  $A$ -module with the action  $aw = c(a)w$  for all  $a \in A$ . Then  $W$  has character  $c|_A$ . We form the  $c$ -induced module  $V = u(L, c) \otimes_{u(A, c|_A)} W$ . See [4, Chapter 5]. By [4, Proposition 5.6.2],  $\dim(V) = p^{\dim(L/A)} \leq p^{n-1}$ . Let  $V_0$  be the direct sum of the composition factors of  $V$ . Then  $\dim(V_0) \leq p^{n-1}$ . Note that  $A$  acts non-trivially on  $V_0$  since it acts non-trivially on the irreducible  $A$ -submodule  $1 \otimes W$  of  $V$ . If  $V_0$  is faithful, the result holds.

Let  $\{e_1, \dots, e_k\}$  be a co-basis of  $A$  in  $L$ . Then by [4, Proposition 5.6.2], the  $e_1^{r_1} e_2^{r_2} \dots e_k^{r_k} \otimes w$  with the  $r_i < p$  form a basis of  $V$ . For  $x = \sum \lambda_i e_i + a$  with  $a \in A$ ,  $x(1 \otimes w) = \sum \lambda_i e_i \otimes w + 1 \otimes aw$ . If  $x(1 \otimes w) = 0$  then we must have  $\lambda_i = 0$  for all  $i$ , that is,  $x \in A$ . Thus the representation of  $L$  on  $V$  has kernel  $\ker(V) \subseteq A$ . As  $A$  is a minimal ideal and acts non-trivially, we have  $\ker(V) = 0$ . Thus  $V$  is faithful.

So suppose that  $V_0$  is not faithful. Then there exists a minimal ideal  $B$  whose action on every composition factor of  $V$  is trivial. Then  $B$  is represented on  $V$  by nilpotent linear transformations. But  $V$  is faithful, so by Engel's Theorem for algebras of linear transformations,  $B$  is nilpotent. But  $B'$  is an ideal of  $L$ , so we must have  $B' = 0$ . By our choice of  $[p]$ ,  $B$  is a  $[p]$ -ideal of  $L$ , contrary to  $A$  being the only minimal  $[p]$ -ideal of  $L$ . Therefore  $V$  is faithful.  $\square$

### 3. MINIMAL $p$ -ENVELOPES.

Let  $(L^e, [p])$  be a minimal  $p$ -envelope of  $L$ . We investigate  $\dim(L^e)$ . Note that, by [4, Theorem 2.5.8(1)],  $\dim(L^e)$  is independent of the choice of minimal  $p$ -envelope. Let  $Z$  be the centre of  $L^e$ . By [4, Theorem 2.5.8(3)],  $Z \subseteq L$ . By [4, Proposition 2.1.3(2)],  $(L^e)' \subseteq L$ .

**Lemma 3.1.** *Let  $A$  be an ideal of  $L$ . Then  $A$  is an ideal of  $L^e$ .*

*Proof.* The set  $\{x \in L^e \mid \text{ad}(x)A \subseteq A\}$  is a  $[p]$ -subalgebra of  $L^e$  and contains  $L$ .  $\square$

**Lemma 3.2.** *Let  $a_1, \dots, a_r \in L^e$  and  $\lambda_1, \dots, \lambda_r \in F$ . Then*

$$\left(\sum_{i=1}^r \lambda_i a_i\right)^{[p]} = \sum_{i=1}^r \lambda_i^p a_i^{[p]} + k$$

for some  $k \in L$ .

*Proof.* From the definition of a  $p$ -operation, we have  $(\lambda_i a_i)^{[p]} = \lambda_i^p a_i^{[p]}$ . The result holds for  $r = 2$  by Lemma 2.1(3) since  $(L^e)' \subseteq L$ . So  $(\lambda_1 a_1 + \dots + \lambda_r a_r)^{[p]} = (\lambda_1 a_1 + \dots + \lambda_{r-1} a_{r-1})^{[p]} + \lambda_r^p a_r^{[p]} + k_1$  for some  $k_1 \in L$ . But by induction,  $(\lambda_1 a_1 + \dots + \lambda_{r-1} a_{r-1})^{[p]} = \lambda_1^p a_1^{[p]} + \dots + \lambda_{r-1}^p a_{r-1}^{[p]} + k_2$  for some  $k_2 \in L$ . The result follows.  $\square$

**Lemma 3.3.** *Let  $x \in L$  and let  $V = \langle x^{[p]^i} \mid i = 1, 2, \dots \rangle$  be the space spanned by the  $x^{[p]^i}$ . Then  $\dim((V + L)/L) \leq n$ .*

*Proof.* We have  $\text{ad}(x)L^e \subseteq L$ . The maps  $\text{ad}(x^{[p]^i})|_L \rightarrow L$  are powers of  $\text{ad}(x)|_L$ . So they span a subspace of  $\text{Hom}(L, L)$  of dimension at most  $n$ . For some  $r \leq n-1$ , the maps  $\text{ad}(x)|_L, \text{ad}(x^{[p]})|_L, \dots, \text{ad}(x^{[p]^r})|_L$  are linearly independent with

$$\text{ad}(x^{[p]^{r+1}})|_L = \sum_{i=0}^r \lambda_i \text{ad}(x^{[p]^i})|_L$$

for some  $\lambda_i \in F$ . Put  $y = x^{[p]^{r+1}} - \sum_{i=0}^r \lambda_i x^{[p]^i}$ . Then  $\text{ad}(y)L^e \subseteq L$  and  $\text{ad}(y)L = 0$ . Thus  $\text{ad}(y)^p L^e = 0$  and it follows that  $y^{[p]} \in Z \subseteq L$ .

By Lemma 2.1(1) and (2),  $y^{[p]} = x^{[p]^{r+2}} - \sum_{i=0}^r \lambda_i^p x^{[p]^{i+1}}$ . Thus  $x^{[p]^{r+2}} \in \langle x^{[p]}, \dots, x^{[p]^{r+1}} \rangle + Z$ . Suppose that  $x^{[p]^{r+s}} \in \langle x^{[p]}, \dots, x^{[p]^{r+1}} \rangle + Z$ . Then  $x^{[p]^{r+s}} = \mu_1 x^{[p]} + \dots + \mu_{r+1} x^{[p]^{r+1}} + z$  for some  $\mu_i \in F$  and  $z \in Z$ . By Lemma 2.1(1) and (2),  $x^{[p]^{r+s+1}} = \mu_1^p x^{[p]^2} + \dots + \mu_{r+1}^p x^{[p]^{r+2}} + z^{[p]}$ . Since  $x^{[p]^{r+2}} \in \langle x^{[p]}, \dots, x^{[p]^{r+1}} \rangle + Z$  and  $z^{[p]} \in Z$ , we have  $x^{[p]^{r+s+1}} \in \langle x^{[p]}, \dots, x^{[p]^{r+1}} \rangle + Z$ . It follows by induction over  $s$ , that  $\langle x^{[p]}, \dots, x^{[p]^{r+1}} \rangle + Z = V + Z$  and so, that  $\dim((V+L)/L) \leq r+1 \leq n$ .  $\square$

**Theorem 3.4.** *Let  $L$  be a Lie algebra of dimension  $n$  over the field  $F$  of characteristic  $p > 0$  and let  $A$  be an abelian ideal of  $L$  with  $\dim(A) = d$ . Let  $(L^e, [p])$  be a minimal  $p$ -envelope of  $L$ . Then  $\dim(L^e) \leq n(n-d+1)$ .*

*Proof.* We choose a basis  $\{e_1, \dots, e_n\}$  of  $L$  with  $e_{n-d+1}, \dots, e_n \in A$ . By Lemma 2.2, we may suppose that  $a^{[p]} = 0$  for all  $a \in A$ . Then the  $e_i^{[p]^j} = 0$  for  $i > n-d$  and  $j > 0$ . For each  $i$ , let  $V_i$  be the subspace of  $L^e$  spanned by the  $e_i^{[p]^j}$  (including  $j = 0$ ) and let  $V = \sum_i V_i$ . Then  $V \supseteq L$  and  $V/L = \sum_{i=1}^{n-d} (V_i + L)/L$ . By Lemma 3.3,  $\dim(V_i + L/L) \leq n$ , so  $\dim(V/L) \leq n(n-d)$  giving  $\dim(V) \leq n(n-d+1)$ . But  $(L^e)' \subseteq L$ , so  $V$  is a subalgebra of  $L^e$ . By Lemma 3.2,  $v^{[p]} \in V$  for all  $v \in V$ . Thus  $V$  is a  $p$ -envelope of  $L$ , so  $V = L^e$ .  $\square$

#### 4. THE MAIN RESULT.

*Proof of Theorem 1.1.* We use induction over  $n$ . Suppose that  $A_1, A_2$  are distinct minimal ideals of  $L$ . Then  $L/A_i$  has a faithful completely reducible module  $V_i$  with  $\dim(V_i) \leq p^{(n-1)^2-1}$  and  $V_1 \oplus V_2$  is a module satisfying all the requirements. So suppose that  $A$  is the only minimal ideal of  $L$ . If  $A$  is non-abelian, then  $A$  is an  $L$ -module satisfying the requirements, so suppose that  $A$  is abelian.

We take a minimal  $p$ -envelope  $(L^e, [p])$  of  $L$ . As  $\dim(A) \geq 1$ , by Theorem 3.4, we have  $\dim(L^e) \leq n^2$ . By Theorem 2.4,  $L^e$  has a faithful completely reducible module  $V$  with  $\dim(V) \leq p^{n^2-1}$ . There is some irreducible summand  $V_0$  of  $V$  on which  $A$  acts non-trivially. By Lemma 3.1,  $A$  is an ideal of  $L^e$  and it follows that  $V_0^A := \{v \in V_0 \mid Av = 0\}$  is an  $L^e$ -submodule of  $V_0$ . Therefore  $V_0^A = 0$ . Let  $V_1$  be an irreducible  $L$ -submodule of  $V_0$ . Since  $V_1^A \subseteq V_0^A$ , we have  $V_1^A = 0$ . But  $A$  is the only minimal ideal of  $L$ . As it is not in the kernel of the representation of  $L$  on  $V_1$ ,  $V_1$  is a faithful  $L$ -module.  $\square$

**Remark 4.1.** We have a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ , namely  $f(n) = p^{n^2-1}$ , such that every Lie algebra of dimension  $n$  over a field of characteristic  $p$  has a faithful completely reducible module of dimension less than or equal to  $f(n)$ . We cannot replace this with a function independent of  $p$ , for suppose that  $f : \mathbb{N} \rightarrow \mathbb{N}$  were claimed to be such a function. The smallest faithful completely reducible module for the non-abelian algebra of dimension 2 has dimension  $p$ , so this algebra over a field of characteristic  $p > f(2)$  is a counterexample. This does not rule out the possibility, if we drop the requirement of complete reducibility, of there being a function  $f$  independent of  $p$  such that every Lie algebra of dimension  $n$  over a field of non-zero characteristic has a faithful module of dimension less than or equal to  $f(n)$ .

It is not claimed that any of the bounds given in this paper are best possible.

# REFERENCES

1. D. W. Barnes, *Ado-Iwasawa extras*, J. Aust. Math. Soc. **78** (2005), 407–421.
2. D. W. Barnes, *Faithful representations of Leibniz algebras*, Proc. Amer. Math. Soc. **141** (2013), 2991–2995. Also arXiv:1111.2627.
3. G. Hochschild, *An addition to Ado's Theorem*, Proc. Amer. Math. Soc. **17** (1966), 531–533.
4. H. Strade and R. Farnsteiner, *Modular Lie algebras and their representations*, Marcel Dekker, Inc., New York-Basel, 1988.

1 LITTLE WONGA RD., CREMORNE NSW 2090, AUSTRALIA,  
*E-mail address:* D.Barnes@maths.usyd.edu.au